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A factorization property of the Berry phase

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Abstract. It is shown that for systems with Hamiltonians expressible in terms of the generators of a Lie group, the Berry phase appears in the form of a product of a representation dependent term and a purely geometrical factor.

In several soluble systems of physical interest the Hamiltonian can be expressed as $\sum a_i T_i$, where T_i are the generators of a Lie group. Thus in the two commonly cited examples exhibiting the Berry phase, namely, the generalized harmonic oscillator and the spin in a magnetic field the Hamiltonian can be written in terms of the generators of $SO(2,1) \approx SU(1,1)$ and $SU(2)$ respectively. In both these cases the Berry phase for any stationary state appears in the product form

$$\Phi = fg. \tag{1}$$

The term f brings in the dependence on the quantum number of the state being adiabatically transported while the term g is purely geometrical and independent of representation. Our purpose here is to show that whenever the Hamiltonian can be written in terms of the generators of a Lie group, such factorization is a natural consequence.

The Berry phase, as originally formulated, needed the adiabatic theorem to ensure that the transported state returns to itself (but for a phase) so that the evolution in ray space is periodic. It is now well known [3, 4] that this phase, also called the geometric phase, depends only on the closed curve in the projective Hilbert space of rays and can be formulated entirely in terms of geometric structures on this space.

Let us, therefore, consider the irreducible unitary representation $T(g)$ of a Lie group G acting on the Hilbert space $\mathcal{H}$. A smooth parametrization of $\mathcal{H}$ can then be achieved through the generalised coherent states [5–8]. These are states obtained by the action of $T(g)$ on some standard state Ψ_0. If $T(h)$, $h \in H \subset G$, form a representation of the stationary subgroup of Ψ_0, i.e. $T(h)\Psi_0 = e^{i\phi(h)}\Psi_0$, then $T(gh)\Psi_0$ and $T(g)\Psi_0$ differ only by a phase and therefore coherent states are in one to one correspondence with the points in the coset space G/H thereby providing a natural parametrization of $\mathcal{H}$. The standard state can be any state in the representation space but for convenience it is taken to be the state of lowest (or highest) weight.

Let us consider a three element Lie algebra:

$$[H, E_\pm] = \pm E_\pm \qquad [E_+, E_-] = fH. \tag{2}$$

The lowest state being Ψ_0 with $H\Psi_0 = h_0\Psi_0$, coherent states are constructed as,

$$\Psi(\eta) = D(\eta)\Psi_0 = \exp(\eta E_+ - \eta^* E_-)\Psi_0. \tag{3}$$

The operators $D(\eta)$ are unitary but do not form a group. However for any η, η' there exists a γ and some function $\phi(H)$ such that $D(\eta)D(\eta') = D(\gamma)\mathrm{e}^{\mathrm{i}\phi(H)}$, showing that $D(\eta)$ is a projective representation of the factor group G/H and the variables $\mathrm{Re}\,\eta$ and $\mathrm{Im}\,\eta$ can be taken to parametrize the Hilbert space.

In the general case, with the basis $(H_i, E_{\pm\alpha})$, and H_i spanning the Cartan subalgebra, the coherent states will be obtained as

$$\Psi(\eta_\alpha) = \exp\sum[\eta_\alpha E_\alpha - \eta_\alpha^* E_{-\alpha}]\Psi_0 \tag{4}$$

the state Ψ_0, for convenience, again being the state of lowest weight. However, in this case Ψ_0 may in general be destroyed by some of the operators E_α and such E_α and their conjugates shall be excluded from the sum in equation(4) (for details see Zhang [8]).

When the standard state is the lowest weight state, the construction of the coherent state is facilitated by the so called Gaussian decomposition, in the normal order:

$$D(\eta_\alpha) = \exp\sum_\alpha Z_\alpha E_\alpha \cdot \exp\sum_j a_j H_j \cdot \exp\sum_\alpha Z'_\alpha E_{-\alpha}. \tag{5}$$

A similar decomposition in the anti-normal order is also possible in which case the highest weight state becomes suitable.

Given a Hilbert space of normalised state vectors $\Psi(\eta)$, $\eta \equiv (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$, one constructs the following symplectic two-form [9]

$$\sigma(\eta) \equiv \mathrm{Im}(\mathrm{d}\Psi(\eta), \mathrm{d}\Psi(\eta)) = \mathrm{Im}\left(\frac{\partial\Psi(\eta)}{\partial\eta_i}, \frac{\partial\Psi(\eta)}{\partial\eta_j}\right) \mathrm{d}\eta_i \wedge \mathrm{d}\eta_j \equiv \sigma_{ij}\mathrm{d}\eta_i \wedge \mathrm{d}\eta_j \tag{6}$$

where (,) denotes an inner product and d stands for exterior differentiation.

The remarkable property of σ is that it is a ray space object i.e. σ is the same for $\Psi(\eta)$ and $\Psi(\eta)\mathrm{e}^{\mathrm{i}\phi(\eta)}$. This is easily seen in the following way. Locally σ can be written as $\sigma = \mathrm{d}\beta$ where $\beta = -\mathrm{i}(\Psi(\eta), \mathrm{d}\Psi(\eta))$. Under a local gauge transformation $\Psi(\eta) \to \Psi'(\eta) = \Psi(\eta)\mathrm{e}^{\mathrm{i}\phi(\eta)}$, β transforms as $\beta \to \beta' = \beta + \mathrm{d}\phi$ and since $\mathrm{d}(\mathrm{d}\phi) = 0$, σ is invariant. By construction σ is also reparametrization invariant. In a two parameter Hilbert space, σ has the meaning of a signed area element in ray space. The geometric phase for a closed curve C in ray space is given by

$$\Phi(C) = \int_{c=\partial s} \beta = \int_s \mathrm{d}\beta = \int_s \sigma\,. \tag{7}$$

We shall now show a factorization property of σ itself which will induce the required factorization property of Φ. We denote by $\eta'\eta$ the parameter of the product $D(\eta')D(\eta)$. However, this should cause no confusion. Then since $D(\eta')\Psi(\eta)$

[$= D(\eta')D(\eta)\Psi_0$] and $D(\eta'\eta)\Psi_0$ differ by a phase factor and $D(g)$ is a unitary representation, we have

$$\begin{aligned}\sigma(\eta) &= \mathrm{Im}(\mathrm{d}\Psi(\eta), \mathrm{d}\Psi(\eta)) \\ &= \mathrm{Im}(D(\eta')\mathrm{d}\Psi(\eta), D(\eta')\mathrm{d}\Psi(\eta)) \\ &= \mathrm{Im}(\mathrm{d}[D(\eta')\Psi(\eta)], \mathrm{d}[D(\eta')\Psi(\eta)]) \\ &= \mathrm{Im}(\mathrm{d}\Psi(\eta'\eta), \mathrm{d}\Psi(\eta'\eta)) = \sigma(\eta'\eta).\end{aligned} \tag{8}$$

Thus σ is left-invariant and therefore its components can be calculated at any suitable point and then by translation determined elsewhere.

We choose to determine σ_{ij} in the neighbourhood of the standard state Ψ_0 itself whose parameter values,by construction, are zero, around which point

$$\begin{aligned}&\Psi(\eta) \approx (1 + \eta E_+)\Psi_0 \\ &\left.\frac{\partial\Psi(\eta)}{\partial\eta_1}\right|_{\eta=0} = E_+\Psi_0 \qquad \left.\frac{\partial\Psi(\eta)}{\partial\eta_2}\right|_{\eta=0} = \mathrm{i}E_+\Psi_0 \\ &\sigma(\eta) = -2fh_0\,\mathrm{d}\eta_1 \wedge \mathrm{d}\eta_2\end{aligned} \tag{9}$$

in the neighbourhood of $\eta = 0$.

Thus the representation dependent part fh_0 is factored out in the neighbourhood of the standard state. Translation to any finite η then involves the Jacobian, $J = \det \mid (\partial(\eta\eta')/(\partial\eta') \mid_{\eta'=0}$. The form of the product parameter $\eta\eta'$ as a function of η' is independent of representation and, therefore, the Jacobian is representation independent. Thus quite generally σ has the form,

$$\sigma(\eta) = -\frac{2fh_0}{\det \mid \partial(\eta\eta')/\partial\eta' \mid_{\eta'=0}}\,\mathrm{d}\eta_1 \wedge \mathrm{d}\eta_2. \tag{10}$$

Example 1. Heisenberg–Weyl algebra. Here the relevant commutation relation is $[a, a^\dagger] = 1$ and accordingly the (Glauber) coherent state is given by $\Psi(\eta) = D(\eta)|0\rangle = \exp(\eta a^\dagger - \eta^* a)|0\rangle$. Since $fh_0 = -1$, we have by equation (9) $\sigma(0) = 2\mathrm{d}\eta_1 \wedge \mathrm{d}\eta_2$, and because $D(\eta\eta') = D(\eta + \eta')$ the Jacobian is unity and therefore $\sigma(\eta) = 2\mathrm{d}\eta_1 \wedge \mathrm{d}\eta_2$ everywhere.

Example 2. $SU(2)$ algebra. The relevant commutation relations are $[J_+, J_-] = 2J_0$, $[J_0, J_\pm] = \pm J_\pm$ and accordingly the (angular momentum) coherent state [10, 11] is given by $\Psi(\eta) = D(\eta)|j, -j\rangle = \exp(\eta J_+ - \eta^* J_-)|j - j\rangle$. Since $fh_0 = -2j$, we have by equation (9) $\sigma(0) = 4j\,\mathrm{d}\eta_1 \wedge \mathrm{d}\eta_2$. With $\eta = -\frac{1}{2}\theta\mathrm{e}^{-\mathrm{i}\phi}$, the product rule is given more conveniently in terms of another variable (stereographic projection from a sphere to a tangent plane) $\tau = -\tan(\frac{1}{2}\theta)\mathrm{e}^{-\mathrm{i}\phi}$ which yields $\tau\tau' = (\tau + \tau')/(1 - \tau\tau'^*)$. The Jacobian is then $J = (1 + |\tau|^2)^2$, and therefore

$$\sigma(\tau) = \frac{4j}{(1 + |\tau|^2)^2}\,\mathrm{d}\tau_1 \wedge \mathrm{d}\tau_2 = j\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi.$$

Example 3. $SU(1,1)$ algebra. In this case, the relevant commutation relations are $[K_+, K_-] = -2K_0$ and $[K_0, K_\pm] = \pm K_\pm$. and the corresponding coherent state is given by $\Psi(\eta) = D(\eta)|k, k\rangle = \exp(\eta K_+ - \eta^* K_-)|k, k\rangle$, $k = 1, \frac{3}{2}, 2, \ldots$.

Here $fh_0 = 2k$ and $\sigma(0) = 4k\,\mathrm{d}\eta_1 \wedge \mathrm{d}\eta_2$. Writing $\eta = -\frac{1}{2}\theta e^{i\phi}$ and introducing $\tau = -\tanh\frac{1}{2}\theta e^{i\phi}$ (stereographic projection from a hyperboloid on to a plane), we have the product rule, $\tau\tau' = (\tau + \tau')/(1 + \tau^*\tau')$. The Jacobian is $J = (1 - |\tau|^2)^2$ and

$$\sigma(\tau) = \frac{4k}{(1 - |\tau|^2)^2}\,\mathrm{d}\tau_1 \wedge \mathrm{d}\tau_2 = k \sinh\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi.$$

Let us now return to the general case given by equation (4). With $[E_\alpha, E_{-\alpha}] = \alpha_i H_i$, we find that in the neighbourhood of $\eta_\alpha = 0$, σ has the form

$$\sigma = \mathrm{Im} \sum_{\alpha,\beta} (\Psi_0, E_{-\alpha} E_\beta \Psi_0)\mathrm{d}\eta^*_\alpha \wedge \mathrm{d}\eta_\beta$$

the sum running over the allowed α's as discussed. Each term in σ upon translation gets multiplied by the appropriate Jacobian and σ in general has the form of a sum of factorised terms. The whole argument could be repeated with any other state in the representation as the standard state. This will only change the eigenvalues of the Cartan operators. Going back to the Hamiltonian, we observe that whenever the Hamiltonian can be written in terms of Lie group generators, its eigenstates will belong to the representation space of the group and any of them could be chosen to be the standard state. All the foregoing arguments then apply and the Berry phase appears in a factorised form as we have set out to prove.

We want, now, to make a few additional remarks. One can introduce a metric [9] in Hilbert space through the following gauge and reparametrization invariant distance function $D(\bar{\Psi}_1, \bar{\Psi}_2)$:

$$D^2(\bar{\Psi}_1, \bar{\Psi}_2) = \inf_{\delta_1,\delta_2} |\, \Psi_1 e^{i\delta_1} - \Psi_2 e^{i\delta_2} \,|^2 = 2 - 2\,|\,\langle \Psi_1, \Psi_2\rangle\,|^2 .$$

This induces the following metric:

$$D^2(\bar{\Psi}(\eta + \mathrm{d}\eta), \bar{\Psi}(\eta)) = g_{ij}(\eta)\,\mathrm{d}\eta_i\,\mathrm{d}\eta_j .$$

A unitary operator is therefore an isometry, i.e.

$$D^2(\widetilde{D\Psi}_1, \widetilde{D\Psi}_2) = D^2(\bar{\Psi}_1, \bar{\Psi}_2)$$

and we find the metric to be left-invariant

$$g_{ij}(\eta)\mathrm{d}\eta_i \mathrm{d}\eta_j = g_{ij}(\beta\eta)\mathrm{d}(\beta\eta)_i \mathrm{d}(\beta\eta)_j .$$

Thus, the metric also can be calculated at a suitable point and by translation determined elsewhere. Near $\eta = 0$, the metric is given by

$$g_{ij}\mathrm{d}\eta_i \wedge \mathrm{d}\eta_j = -fh_0(\mathrm{d}\eta_1^2 + \mathrm{d}\eta_2^2)$$

which also is in a factorised form and will remain so under translation. Again with the standard state as any other state in the representation, only the eigenvalues of the Cartan operators will change; the geometric part being representation independent will remain the same.

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